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PARTIAL EIGENSTRUCTURE ASSIGNMENT FOR THE QUADRATIC PENCIL

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It is shown in this paper that, by the appropriate choice of gain and input influence matrices, certain eigenpairs of a vibrating system may be assigned while the other eigenpairs remain unchanged. The system under consideration is modelled by a set of second order differential equations and the assignment is carried by multi-input state feedback control. The solution may be of particular interest in the stabilization and control of flexible structures using smart materials, where only a small part of the eigenstructure is to be reassigned and the rest is required to remain unchanged. The method presented is illustrated with a numerical example.

1. INTRODUCTION

Consider the vibratory system modelled by the second order matrix differential equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \tag{1}$$

where the dots denote differentiation with respect to time and the n-square real matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric. Separation of variables

$$\mathbf{x}(t) = \mathbf{z}e^{\lambda t}, \quad \mathbf{z} \text{ a constant vector}$$

in equation (1), leads to the quadratic eigenvalue problem of finding the eigenvalues λ_k and the associated eigenvectors $\mathbf{z}_k \neq 0$, which satisfy

$$P(\lambda_k)\mathbf{z}_k = \mathbf{0}, \quad k = 1, 2, \ldots, 2n, \tag{2}$$

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where

$$P(\lambda) = (\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}).$$

Assembling the $2n$ relations (2) we can write

$$\mathbf{MZ}\boldsymbol{\Lambda}^2 + \mathbf{CZ}\boldsymbol{\Lambda} + \mathbf{KZ} = \mathbf{O},$$

where $\boldsymbol{\Lambda} = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{2n}\}$ and $\mathbf{Z} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{2n})$. Our interest here is in the case where the set $\{\lambda_k\}_1^{2n}$ is distinct, from which it follows that the eigenvectors $\{\mathbf{z}_k\}_1^{2n}$ are two-fold linearly independent in the sense that

$$\mathbf{W} = \begin{pmatrix} \mathbf{Z} \\ \mathbf{Z}\boldsymbol{\Lambda} \end{pmatrix} \tag{3}$$

is invertible. If $(\lambda, \mathbf{z})$ is an eigenpair of equation (2) then the complex conjugate $(\bar{\lambda}, \bar{\mathbf{z}})$ is also an eigenpair because $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real. Hence, we can say that the sets $\{\lambda_k\}_1^{2n}$ and $\{\mathbf{z}_k\}_1^{2n}$ are *pairwise self-conjugate* in the sense that they are self-conjugate and $\mathbf{z}_p = \bar{\mathbf{z}}_q$ whenever $\lambda_p = \bar{\lambda}_q$, for all p and q. Where there is no ambiguity, we will refer to a diagonal matrix of the λ_k and the matrix of corresponding $\mathbf{z}_k$ as pairwise self-conjugate if the associated sets are pairwise self-conjugate.

The dynamics of equation (1) can be modified by applying a control force $\mathbf{Bu}(t)$, $\mathbf{B}$ an $n \times m$ matrix and $\mathbf{u}(t)$ a time-dependent m vector. The model relation (1) now becomes

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{Kx} = \mathbf{Bu}(t). \tag{4}$$

The special choice

$$\mathbf{u}(t) = \mathbf{F}^{\mathrm{T}}\dot{\mathbf{x}} + \mathbf{G}^{\mathrm{T}}\mathbf{x}, \tag{5}$$

where $\mathbf{F}$ and $\mathbf{G}$ are $n \times m$ matrices, is called *state feedback control* and leads to the eigenvalue problem

$$\mathbf{MYD}^2 + (\mathbf{C} - \mathbf{BF}^{\mathrm{T}})\mathbf{YD} + (\mathbf{K} - \mathbf{BG}^{\mathrm{T}})\mathbf{Y} = \mathbf{O}, \tag{6}$$

where $\mathbf{Y} \in \mathscr{C}^{n \times 2n}$ is the eigenvector matrix and the diagonal $D \in \mathscr{C}^{2n \times 2n}$ is the eigenvalue matrix.

We note in passing that whereas equation (5) applies state feedback control using position and velocity, the choice

$$\mathbf{u}(t) = \mathbf{F}^{\mathrm{T}}\ddot{\mathbf{x}} + \mathbf{G}^{\mathrm{T}}\dot{\mathbf{x}}$$

applies state feedback control using acceleration and velocity. This choice leads to a problem which can be recast as a position and velocity problem for the same $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ matrices but taken in the reverse order. We leave the details to the interested reader.

The problem of finding $\mathbf{F}$ and $\mathbf{G}$ such that the closed-loop quadratic pencil $\lambda^2\mathbf{M} + \lambda(\mathbf{C} - \mathbf{BF}^{\mathrm{T}}) + (\mathbf{K} - \mathbf{BG}^{\mathrm{T}})$ has a desired set of $2n$ eigenvalues is called the eigenvalue assignment or more popularly, the pole placement problem, in control theory literature. In most practical situations, however, only a few eigenvalues of the open-loop pencil $P(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}$ are undesirable (i.e., do not lie in the left-half plane as required for stability). In those situations, it makes more sense to replace only the undesirable eigenvalues while leaving the others unchanged. This modified pole placement problem is called *the partial pole placement problem.* The partial pole placement problem for the quadratic pencil $P(\lambda)$ has been solved recently in the single- and multi-input cases [1, 2]. The solutions in both cases have been obtained solely in the second order setting in the sense that they do not depend on a first order realization [3, 4] and deal directly with matrices **M**, **K** and **C**. While the pole placement problem is important in its own right, it is to be noted that, if the system transient response needs to be altered by feedback, both eigenvalue placement as well as eigenvector placement should be considered.

This is easily seen from the model expansion theorem (see references [3, 5]) which says that every solution $x(t)$ of equation (1) in the form $x(t) = \mathbf{z}e^{\lambda t}$, representing a free response of equation (1), can be written in terms of the eigenvalues and eigenvectors of the pencil $P(\lambda)$:

$$x(t) = \sum_{k=1}^{2n} a_k e^{\lambda_k t} \mathbf{z}_k.$$

Thus, the eigenvalues determine the rate at which the system response decays or grows, while the eigenvectors determine the shape of the response.

The problem of altering both the eigenvalues and the eigenvectors of the closed-loop pencil is known as the *eigenstructure assignment problem.*

For the second order system eigenstructure problem, see References [3, 6–8] and for the first order system, see references [9–12].

Unfortunately, the eigenstructure problem, in general, is not solvable if the matrix **B** is given (see reference [6]). Recent progress with smart materials makes the concept of full-state feedback, with a dense matrix **B**, possible [13] and practical. Also, control of robot vibration allows application of a full-state feedback control. In this paper we consider a more tractable problem, namely the partial eigenstructure assignment problem by allowing **B** to be chosen. Specifically, we consider Problem 1.1. stated below and obtain a solution of the problem entirely in the second order setting, without resorting to the first order realization, so that the problem order is not doubled, the inverse of **M** is not computed explicitly and the exploitable structures offered by the problem, such as sparsity, symmetry, definiteness, etc., are preserved.

In order that the control be realizable by means of physical devices, the matrices **B**, **F** and **G** must all be real. In such a case, the eigenvalues and eigenvectors are pairwise self-conjugate.

Let us partition the $n \times 2n$ eigenvector matrix and $2n \times 2n$ eigenvalue matrix as follows:

$$\mathbf{Z} = (\underset{m}{\mathbf{Z}_1} \quad \underset{2n-m}{\mathbf{Z}_2}), \quad \mathbf{\Lambda} = \underset{\begin{matrix} m & 2n-m \end{matrix}}{\begin{pmatrix} \mathbf{\Lambda}_1 & \\ & \mathbf{\Lambda}_2 \end{pmatrix}} \begin{matrix} m \\ 2n-m \end{matrix},$$

where $\mathbf{Z}_1$ and $\mathbf{\Lambda}_1$ are pairwise self-conjugate.

In this paper we address the following.

Problem 1.1. *Given*

(a) *real symmetric* $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$,

(b) $\mathbf{Z}_1$ *and* $\mathbf{\Lambda}_1$ *pairwise self-conjugate,*

(c) $\mathbf{Y}_1 \in \mathscr{C}^{n\times m}$, $\mathbf{D}_1 \in \mathscr{C}^{m\times m,}$ *pairwise self-conjugate such that with*

$$\mathbf{Y} = \underset{m\quad\ 2n-m}{(\mathbf{Y}_1 \quad \mathbf{Z}_2)}, \qquad \mathbf{D} = \underset{m \qquad 2n-m}{\begin{pmatrix} \mathbf{D}_1 & \\ & \mathbf{\Lambda}_2 \end{pmatrix}} \begin{matrix} m \\ 2n-m \end{matrix},$$

the matrix

$$\begin{pmatrix} \mathbf{Y} \\ \mathbf{YD} \end{pmatrix} \tag{7}$$

is invertible,

find B, F, $G \in \mathscr{R}^{n\times m}$ *such that equation* (6) *holds.*

2. MAIN RESULTS

The solution process consists of two stages:

(a) Determine matrices $\hat{\mathbf{B}}$, $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ which are generally complex and which satisfy

$$\mathbf{MYD}^2 + (\mathbf{C} - \hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}})\mathbf{YD} + (\mathbf{K} - \hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}})\mathbf{Y} = \mathbf{O}. \tag{8}$$

(b) From $\hat{\mathbf{B}}$, $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ find real $\mathbf{B}$, $\mathbf{F}$, and $\mathbf{G}$ such that $\mathbf{BF}^{\mathrm{T}} = \hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}$ and $\mathbf{BG}^{\mathrm{T}} = \hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}$.

Let us focus first on stage (a).

Suppose that $\tilde{\mathbf{B}}$, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ be a solution. Then

$$\mathbf{MY}_1\mathbf{D}_1^2 + \mathbf{CY}_1\mathbf{D}_1 + \mathbf{KY}_1 = \tilde{\mathbf{B}}(\tilde{\mathbf{F}}^{\mathrm{T}}\mathbf{Y}_1\mathbf{D}_1 + \tilde{\mathbf{G}}^{\mathrm{T}}\mathbf{Y}_1). \tag{9}$$

Suppose that $\tilde{\mathbf{B}}$, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ is a solution to Problem 1.1, and let $\mathbf{W} \in \mathscr{C}^{m\times p}$, $p \geqslant m$ have pseudoinverse $W^+ \in \mathscr{C}^{p\times m}$ such that $\mathbf{WW}^+ = \mathbf{I} \in \mathscr{R}^{m\times m}$. Then $\hat{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{W}$, $\hat{\mathbf{F}} = \tilde{\mathbf{F}}\mathbf{W}^+$ and $\hat{\mathbf{G}} = \grave{\mathbf{G}}\mathbf{W}^+$, is another solution because $\tilde{\mathbf{B}}\tilde{\mathbf{F}}^{\mathrm{T}} = \hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}$ and $\tilde{\mathbf{B}}\tilde{\mathbf{G}}^{\mathrm{T}} = \hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}$.

Using $\mathbf{W} \in \mathscr{C}^{m\times p}$ with $p > m$ allows for the construction of a solution in which $\mathbf{B}$ can have dimension $n \times p$, $p > m$. This fact is a consequence of the arbitrariness in the solution which we will not pursue here.

Denote

$$\mathbf{W} = \tilde{\mathbf{F}}^{\mathrm{T}}\mathbf{Y}_1\mathbf{D}_1 + \tilde{\mathbf{G}}^{\mathrm{T}}\mathbf{Y}_1. \tag{10}$$

Then, provided that $\mathbf{W}$ is invertible, $\hat{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{W}$ is admissable for some $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$. We can therefore take

$$\hat{\mathbf{B}} = \mathbf{MY}_1\mathbf{D}_1^2 + \mathbf{CY}_1\mathbf{D}_1 + \mathbf{KY}_1 \tag{11}$$

by virtue of equations (9) and (10). Relations (11) and (8) together imply that

$$\hat{\mathbf{F}}^{\mathrm{T}}\mathbf{Y}_1\mathbf{D}_1 + \hat{\mathbf{G}}^{\mathrm{T}}\mathbf{Y}_1 = \mathbf{I}. \tag{12}$$

In reference [14] it is shown that

Theorem 2.1. *For any $\Phi \in \mathscr{C}^{m\times m}$,*

$$\hat{\mathbf{F}} = \mathbf{M}\mathbf{Z}_1\mathbf{\Lambda}_1\Phi, \qquad \hat{\mathbf{G}} = -\mathbf{K}\mathbf{Z}_1\Phi \tag{13}$$

satisfy

$$\mathbf{M}\mathbf{Z}_2\mathbf{\Lambda}_2^2 + (\mathbf{C} - \hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}})\mathbf{Z}_2\mathbf{\Lambda}_2 + (\mathbf{K} - \hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}})\mathbf{Z}_2 = \mathbf{O}.$$

In other words, $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ of form (13) ensure that the last $2n-m$ eigenpairs of the uncontrolled system are also eigenpairs of the controlled system.

Substituting equation (13) into equation (12) gives

$$\mathbf{\Phi} = (\mathbf{\Lambda}_1\mathbf{Z}_1^{\mathrm{T}}\mathbf{M}\mathbf{Y}_1\mathbf{D}_1 - \mathbf{Z}_1^{\mathrm{T}}\mathbf{K}\mathbf{Y}_1)^{-1} \tag{14}$$

from which $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ can be determined.

The solution $\hat{\mathbf{B}}$, $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ which results from this process is in general complex. However, we now show that the products $\hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}$ and $\hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}$ are always real.

It follows from equation (8) and the pairwise self-conjugacy of $\mathbf{Y}$ and $\mathbf{D}$ that we can write, denoting the conjugates by overbars,

$$\mathbf{M}\overline{\mathbf{Y}\mathbf{D}}^2 + (\mathbf{C} - \hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}})\overline{\mathbf{Y}\mathbf{D}} + (\mathbf{K} - \hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}})\overline{\mathbf{Y}} = \mathbf{O}. \tag{15}$$

Conjugating equation (8) gives

$$\mathbf{M}\overline{\mathbf{Y}\mathbf{D}}^2 + (\mathbf{C} - \overline{\hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}})\overline{\mathbf{Y}\mathbf{D}} + (\mathbf{K} - \overline{\hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}})\overline{\mathbf{Y}} = \mathbf{O}. \tag{16}$$

Subtracting equation (15) from equation (16) gives

$$(\overline{\hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}} - \hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}})\overline{\mathbf{Y}\mathbf{D}} + (\overline{\hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}} - \hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}})\overline{\mathbf{Y}} = \mathbf{O}$$

which can be rewritten in block matrix form as

$$(\overline{\hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}} - \hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}} \,|\, \overline{\hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}} - \hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}})\begin{pmatrix}\overline{\mathbf{Y}}\\ \hline \overline{\mathbf{Y}\mathbf{D}}\end{pmatrix} = \mathbf{O}.$$

The invertibility of equation (7) implies that the left-hand matrix vanishes, from which it follows that $\hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}$ and $\hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}$ are real.

2.1. REAL B, F, AND G FROM $\hat{B}$, $\hat{F}$, AND $\hat{G}$

At the start of the second stage we have generally complex $\hat{\mathbf{B}}$, $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ but real products $\hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}}$ and $\hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}}$. Therefore, let us denote the real $n\times 2n$ product by

$$\mathbf{H} = \hat{\mathbf{B}}[\hat{\mathbf{F}}^{\mathrm{T}} | \hat{\mathbf{G}}^{\mathrm{T}}]$$

and let

$$\mathbf{LR} = \mathbf{H}$$

$\mathbf{L} \in \mathscr{R}^{n \times m}$, $\mathbf{R} \in \mathscr{R}^{m \times 2n}$ be *any* factoring of the right-hand side $\mathbf{H}$. Then we can take $\mathbf{B}$ to be $\mathbf{L}$ and the first n columns of $\mathbf{R}$ to be $\mathbf{F}^{\mathrm{T}}$ and the last n to be $\mathbf{G}^{\mathrm{T}}$.

The two factorings which immediately come to mind for this purpose are the Q factorings and the singular-value decomposition (SVD) (see for example references [15, 16, 5]. We now describe the use of these two factorings to find real $\mathbf{B}$, $\mathbf{F}$, and $\mathbf{G}$. In both of these cases, we use the so-called *truncated* or *compact* form of the factoring.

The truncated QR factorization [15] produces an $\mathbf{L} \in \mathscr{R}^{n \times m}$ in which the m column are orthogonal and an $\mathbf{R} \in \mathscr{R}^{m \times 2n}$ which is upper triangular. For example, in the case of 5×10 matrix $\mathbf{H}$ we have

$$\mathbf{LR} = \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \begin{pmatrix} x & x & x & x & x & | & x & x & x & x & x \\ & x & x & x & x & | & x & x & x & x & x \\ & & x & x & x & | & x & x & x & x & x \end{pmatrix}.$$

By contrast, when the rank of $\mathbf{H}$ is $m \leqslant n$, the compact SVD produces three matrices $\mathbf{U} \in \mathscr{R}^{n \times m}$, orthogonal, $\Sigma \in \mathscr{R}^{m \times m}$, diagonal, and $\mathbf{V} \in \mathscr{R}^{2n \times m}$, orthogonal which are such that

$$\mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}} = \mathbf{H},$$

$$\overset{\mathbf{U}}{\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{pmatrix}} \overset{\Sigma}{\begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix}} \overset{\mathbf{V}^{\mathrm{T}}}{\begin{pmatrix} x & x & x & x & x & | & x & x & x & x & x \\ x & x & x & x & x & | & x & x & x & x & x \\ x & x & x & x & x & | & x & x & x & x & x \end{pmatrix}}.$$

In this case, we take $\mathbf{B}$ to be the product $\mathbf{U}\Sigma$ and we take the first n rows of $\mathbf{V}$ to be $\mathbf{F}$ and the last n rows to be $\mathbf{G}$,

$$\mathbf{LR} = (\mathbf{U}\Sigma)\mathbf{V}^{\mathrm{T}}. \tag{17}$$

3. EXAMPLE

In this section we demonstrate the technique on a simple example. The example models a 4-degree-of-freedom system in which we assign two eigenpairs. The

TABLE 1

Spectra of the open- and closed-loop systems

k	λ_k	d_k
1	$-2{\cdot}0923(e-001)-1{\cdot}8256(e+000)\mathrm{i}$	$-1{\cdot}0000(e+000)-1{\cdot}0000(e+000)\mathrm{i}$
2	$-2{\cdot}0923(e-001)+1{\cdot}8256(e+000)\mathrm{i}$	$-1{\cdot}0000(e+000)+1{\cdot}0000(e+000)\mathrm{i}$
3	$-1{\cdot}3080(e-001)-3{\cdot}1920(e+000)\mathrm{i}$	$-2{\cdot}0923(e-001)-1{\cdot}8256(e+000)\mathrm{i}$
4	$-1{\cdot}3080(e-001)+3{\cdot}1920(e+000)\mathrm{i}$	$-2{\cdot}0923(e-001)+1{\cdot}8256(e+000)\mathrm{i}$
5	$-1{\cdot}2147(e-001)-4{\cdot}4412(e-001)\mathrm{i}$	$-1{\cdot}3080(e-001)+3{\cdot}1920(e+000)\mathrm{i}$
6	$-1{\cdot}2147(e-001)+4{\cdot}4412(e-001)\mathrm{i}$	$-1{\cdot}3080(e-001)-3{\cdot}1920(e+000)\mathrm{i}$
7	$-3{\cdot}8508(e-002)-4{\cdot}1362(e+000)\mathrm{i}$	$-1{\cdot}2147(e-001)+4{\cdot}4412(e-001)\mathrm{i}$
8	$-3{\cdot}8508(e-002)+4{\cdot}1362(e+000)\mathrm{i}$	$-1{\cdot}2147(e-001)-4{\cdot}4412(e-001)\mathrm{i}$

open-loop system we use has the matrices

$$\mathbf{M}=\mathbf{I},\ \mathbf{C}=\operatorname{diag}\{\tfrac{1}{2},0,0,\tfrac{1}{2}\}$$

and

$$\mathbf{K}=\begin{pmatrix} 5 & -5 & 0 & 0\\ -5 & 10 & -5 & 0\\ 0 & -5 & 10 & -5\\ 0 & 0 & -5 & 6\end{pmatrix}.$$

This system has eigenvalues λ_k as shown in Table 1.

We reassign the eigenvalues $\lambda_{7,8}$ and their associated eigenvectors by setting

$$\mathbf{D}_1=\begin{pmatrix}1+\mathrm{i} & \\ & 1-\mathrm{i}\end{pmatrix},\qquad \mathbf{Y}_1=\begin{pmatrix}1+1\mathrm{i} & 1-1\mathrm{i}\\ 1+2\mathrm{i} & 1-2\mathrm{i}\\ 1+3\mathrm{i} & 1-3\mathrm{i}\\ 1+4\mathrm{i} & 1-4\mathrm{i}\end{pmatrix}.$$

Using Theorem 2.1 and equation (14) we get

$$\hat{\mathbf{B}}=\begin{pmatrix}1-7\mathrm{i} & 1+7\mathrm{i}\\ 4-2\mathrm{i} & 4+2\mathrm{i}\\ 6-2\mathrm{i} & 6+2\mathrm{i}\\ 6{\cdot}5+5{\cdot}5\mathrm{i} & 6{\cdot}5-5{\cdot}5\mathrm{i}\end{pmatrix},$$

$$\hat{\mathbf{F}}=\begin{pmatrix}5{\cdot}1427e-001-2{\cdot}4550e-002\mathrm{i} & 5{\cdot}1427e-001+2{\cdot}4550e-002\mathrm{i}\\ -1{\cdot}2016e+000+1{\cdot}1168e-001\mathrm{i} & -1{\cdot}2016e+000-1{\cdot}1168e-001\mathrm{i}\\ 1{\cdot}2253e+000-1{\cdot}1171e-001\mathrm{i} & 1{\cdot}2253e+000+1{\cdot}1171e-001\mathrm{i}\\ -5{\cdot}7169e-001+2{\cdot}4195e-002\mathrm{i} & -5{\cdot}7169e-001-2{\cdot}4195e-002\mathrm{i}\end{pmatrix},$$

$$\hat{\mathbf{G}} = \begin{pmatrix} 7{\cdot}7611e-001+6{\cdot}0498e-001\mathrm{i} & 7{\cdot}7611e-001-6{\cdot}0498e-001\mathrm{i} \\ -2{\cdot}0047e+000-1{\cdot}4635e+000\mathrm{i} & -2{\cdot}0047e+000+1{\cdot}4635e+000\mathrm{i} \\ 2{\cdot}0126e+000+1{\cdot}4914e+000\mathrm{i} & 2{\cdot}0126e-000-1{\cdot}4914e+000\mathrm{i} \\ -8{\cdot}1763e-001-6{\cdot}7288e-001\mathrm{i} & -8{\cdot}1763e-001+6{\cdot}7288e-001\mathrm{i} \end{pmatrix}.$$

However, as mentioned earlier, the products $\hat{B}\hat{F}^{\mathrm{T}}$ and $\hat{B}\hat{G}^{\mathrm{T}}$ are real:

$$\hat{\mathbf{B}}\hat{\mathbf{F}}^{\mathrm{T}} = \begin{pmatrix} 0{\cdot}6848 & -0{\cdot}8398 & 0{\cdot}8867 & -0{\cdot}8047 \\ 4{\cdot}0159 & -9{\cdot}1664 & 9{\cdot}3555 & -4{\cdot}4768 \\ 6{\cdot}0730 & -13{\cdot}9729 & 14{\cdot}2567 & -6{\cdot}7635 \\ 6{\cdot}9555 & -16{\cdot}8497 & 17{\cdot}1576 & -7{\cdot}6982 \end{pmatrix},$$

$$\hat{\mathbf{B}}\hat{\mathbf{G}}^{\mathrm{T}} = \begin{pmatrix} 10{\cdot}0220 & -24{\cdot}4978 & 24{\cdot}9052 & -11{\cdot}0555 \\ 8{\cdot}6288 & -21{\cdot}8914 & 22{\cdot}0663 & -9{\cdot}2326 \\ 11{\cdot}7333 & -29{\cdot}9102 & 30{\cdot}1165 & -12{\cdot}5031 \\ 3{\cdot}4347 & -9{\cdot}9630 & 9{\cdot}7576 & -3{\cdot}2276 \end{pmatrix}.$$

Taking the SVD of $\mathbf{H} = \hat{\mathbf{B}}[\hat{\mathbf{F}}^{\mathrm{T}}|\hat{\mathbf{G}}^{\mathrm{T}}]$ and forming the product in equation (17) gives

$$\mathbf{B} = \begin{pmatrix} 35{\cdot}3526 & 13{\cdot}9956 \\ 36{\cdot}5157 & 0{\cdot}4163 \\ 50{\cdot}7391 & -1{\cdot}5127 \\ 24{\cdot}0369 & -18{\cdot}0236 \end{pmatrix}.$$

Separating the first n and the last n rows of the matrix

$$\mathbf{V} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}$$

in equation (17) yields

$$\mathbf{F} = \begin{pmatrix} 0{\cdot}1127 & -0{\cdot}2357 \\ -0{\cdot}2578 & 0{\cdot}5911 \\ 0{\cdot}2631 & -0{\cdot}6011 \\ -0{\cdot}1256 & 0{\cdot}2597 \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} 0{\cdot}2349 & 0{\cdot}1227 \\ -0{\cdot}5967 & -0{\cdot}2431 \\ 0{\cdot}6013 & 0{\cdot}2606 \\ -0{\cdot}2511 & -0{\cdot}1557 \end{pmatrix}.$$

The eigenvalues of the system controlled by this **B**, **F** and **G** via equation (6) are displayed in Table 1. It can be seen that the assignment of the required eigenvalues has occurred and that eigenvalues intended to remain unchanged are unaltered by the feedback. Although we do not display them, the eigenvectors of the controlled system are assigned as required.

4. CONCLUSION

We have developed a method for the partial eigenstructure assignment of the multi-input state feedback control system modelled by a set of second order differential equations.

We have shown that the input influence matrix **B**, and the gain matrices **F** and **G** can be chosen to assign just a part of the eigenstructure arbitrarily while leaving the rest unchanged. The column dimension of the matrix **B** must be at least as large as the number of eigenpairs to be assigned but **B** can be constructed to have greater column dimension if necessary. But fewer columns cannot achieve the required assignment.

The method developed builds on our previous results in which we determined an explicit solution for the single-input partial pole assignment problem in vibratory systems.

Although the solution here is not unique and is generally complex, we show that, for pairwise self-conjugate data, a real solution is easily available. This is important for practical problems.

The method has been illustrated with a modest numerical example.

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